Homework 1 — Theory

Due: April 26, 2026

Recall: $(f * g)(t) = \int_{-\infty}^{\infty} f(\tau)\, g(t-\tau)\, d\tau$. A system T is **LTI** (linear time-invariant) if $T\{\alpha f + \beta g\} = \alpha\, T\{f\} + \beta\, T\{g\}$ and $T\{f(t-t_0)\} = (Tf)(t-t_0)$.

Prove:

1. **Convolution is commutative:** $f * g \;=\; g * f$
2. **Convolution is associative:** $(f * g) * h \;=\; f * (g * h)$
3. **Convolution is distributive:** $f * (g+h) \;=\; f * g \;+\; f * h$
4. **Delta is the identity:** $f * \delta \;=\; f$
5. **Convolution commutes with differentiation:**
$$\frac{d}{dt}(f * g) \;=\; \frac{df}{dt} * g \;=\; f * \frac{dg}{dt}$$
6. **A system is LTI if and only if it can be expressed by a convolution:** T is LTI $\Leftrightarrow$ $T\{f\} = f * h$ for some kernel h.
   [Hint: $\Leftarrow$ is easy. $\Rightarrow$ express f using δ]
7. **Gaussians convolve into Gaussians:**
$$g_{\sigma_1} * g_{\sigma_2} \;=\; g_{\sqrt{\sigma_1^2+\sigma_2^2}}, \quad \text{where} \quad g_\sigma(t) = \frac{1}{\sigma\sqrt{2\pi}} \exp\!\left(-\frac{t^2}{2\sigma^2}\right)$$
   [Hint: complete the square]
8. **Repeated convolution converges to a Gaussian.**
   Let X_1, X_2 be two independent random variables with PDFs f_1, f_2. Define $S_2 = X_1 + X_2$.
   (a) Write down the joint PDF of (X_1, X_2). What does independence let you do?
   (b) Express the CDF $F_{S_2}(s) = P(S_2 \le s)$ as a double integral over the region $\{x_1 + x_2 \le s\}$, using the joint PDF from (a).
   (c) Differentiate $F_{S_2}(s)$ with respect to s to obtain the PDF $f_{S_2}(s)$. What familiar operation appears?
   (d) Now let $X_1, \ldots, X_n$ be IID with PDF f, mean μ, and finite variance σ^2. Let $S_n = X_1 + \cdots + X_n$. Use the result from (c) to express f_{S_n} in terms of f.
   (e) State the Central Limit Theorem (CLT) for S_n. What does it say about f_{S_n} for large n?
   Recall: the CLT states that $(S_n - n\mu)/(\sigma\sqrt{n}) \to \mathcal{N}(0,1)$ in distribution as $n \to \infty$. See e.g. `en.wikipedia.org/wiki/Central_limit_theorem`.
   (f) Combine (d) and (e): what does this tell you about repeatedly convolving a nonnegative, normalized function with itself?